Homework #9

For Friday, *March 25*

⋆ 1. Consider the equivalence

$$\exists x\ (\varphi(x) \wedge \psi(x)) \leftrightarrow (\exists x\ \varphi(x) \wedge \exists x\ \psi(x)).$$

You can assume that φ and ψ have no free variable other than x.

a. Show that one direction of this equivalence is valid (i.e. true in every structure). Prove this carefully; you can use Lemma 2.4.5.

b. Find examples of φ and ψ where the other direction is not valid (and justify this claim).

⋆ 2. Fix a language, L, which has one binary relation symbol, R. Which of the following statements are true and which are false? Justify your answers.

a. If φ is any sentence, either $\models \varphi$ or $\models \neg\varphi$.

b. If φ is any sentence and $\mathfrak{A}$ is any structure, either $\mathfrak{A} \models \varphi$ or $\mathfrak{A} \models \neg\varphi$.

c. If φ is any sentence and Γ is any set of sentences, then either $\Gamma \models \varphi$ or $\Gamma \models \neg\varphi$.

3. Do problems 6 and 7 on page 72.

4. Do problems 1, 2, and 3 on page 80.

⋆ 5. Do problem 4 on page 80. (Show that for some formula φ in a language of your choosing, the equivalence shown is not valid.)

6. Do problem 5 on page 80. Note that this relies on the convention that we do not consider structures with empty universes.

7. Do problem 6 on page 80.

8. Do problem 12 on page 81. You can use any of the lemmas and theorems in section 2.5 to make your argument as clear as possible. Note that the barber paradox reads as follows: "In a certain town the barber shaves all and exactly those people who do not shave themselves. Who shaves the barber?"

$\star$ 9. Do problem 14c on page 81, assuming φ, ψ, and σ are quantifier-free.

10. Do problem 15 on page 81.

11. Consider the language of orderings with a single binary relation symbol $\leq$. Use $x < y$ as shorthand for $x \leq y \wedge \neg(x = y)$. An element a in a partial ordering $\mathcal{P}$ is *minimal* if there is no element smaller than it; that is, $\mathcal{P} \models \forall x \, \neg(x < \bar{a})$. An element a is *minimum* if it is at least as small as every other element; that is, $\mathcal{P} \models \forall x \, (\bar{a} \leq x)$.

    a. Describe a partial ordering with a minimal element, but no minimum element. (You can use a diagram.)

    b. Show that in a linear ordering, an element is minimum iff it is minimal.

    c. Describe a linear ordering with no minimal element.

    You can argue informally about what is "true in $\mathcal{P}$," without having to use Lemma 2.4.5 explicitly; but your argument should be mathematically rigorous.

12. Do problem 2 on page 90.

$\star$ 13. Do problem 3 on page 90. Help out the grader by explaining, informally, what your formula is supposed to say.

14. Do problem 4 on page 90.

15. Do problem 5 on page 90.

16. Do problem 6 on page 90.

$\star$ 17. Do problem 10 on page 90. This really means, using the language of arithmetic on page 87, define the given relations in the "standard" structure $\langle \mathbb{N}, 0, S, +, \times \rangle$. Note that "$x$ and y are relatively prime" means that they have no common factor other than 1.

18. In $\langle \mathbb{N}, 0, S, +, \times \rangle$, define

    a. the set of primes (i.e. the unary relation "x is a prime")

    b. the set of odd perfect squares

    c. the set of numbers with at least 3 different prime factors

    Use the symbol "0" to denote the element 0, the symbol "S" to denote the successor function S, and so on.

19. In the structure $\langle \mathbb{N}, < \rangle$, define

    a. 0 (i.e. the unary relation $x = 0$)

b. 1

c. The relation "y is the successor of x"

20. Do problem 14 on page 91. Note that the second relation amounts to "reordering" the natural numbers, so that all the even numbers come first, and all the odd ones come next. (This problem is not easy! Give it your best shot.)

∘ 21. Try to define the class of structures with finite universes in the language of equality, or explain why this is impossible. Do the same for well orderings in the language with $\leq$. (In fact, you can add any function or relation symbols you want to these languages.)

∘ 22. If you have some background in abstract algebra, do probems 7, 8, and 9 on page 91.

∘ 23. Do problem 12 on page 91.